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CONDITIONS UNDER WHICH A PENALTY FUNCTION ALGORITHM IS WELL DEFINED

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ABSTRACT:

Conditions are given which ensure that a recently developed exact penalty function algorithm for nonlinear programming problems is well defined.

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I. THE PROBLEM

We consider the nonlinear programming problem with inequality constraints

$$\text{NLP} \quad \text{maximize} \quad f(x)$$
$$\text{subject to} \quad g_i(x) \le 0 \quad i = 1,\dots,m \tag{1}$$
$$x \in E^n$$

where f and g_i are continuously differentiable and for simplicity we assume that a constraint qualification holds. In reference [1] we developed a penalty function algorithm which involves alternate cycles of 1) maximizing a penalty function and 2) approximating optimal Lagrange multipliers. In the interests of brevity the reader is referred to [1] for a complete description of the procedure and its convergence properties. For the purposes of this report it is enough to recall that at each maximization cycle we have fixed $k > 0 \in E^1$, $\lambda > 0 \in E^m$ and we want to find $x \in E^n$ to maximize the penalty function

$$P(x,k,\lambda) = f(x) - (1/k) \sum_{i=1}^{m} \lambda_i [\exp(kg_i(x))-1] \tag{2}$$

In [2] Gould and Howe show that under certain weak conditions if x^* is a local maximum for NLP(1) with optimal Lagrange multipliers λ^*, then there exists a k sufficiently large that $P(x,k,\lambda^*)$ has a local maximum at x^*. Thus, if λ^* is known and k is chosen large enough, the function $P(x,k,\lambda^*)$ is an exact penalty function. The algorithm in [1] attempts to determine the optimal Lagrange multipliers iteratively while simultaneously searching for x^*. Thus, it involves maximizing

$P(x,k,\lambda)$ for fixed $\lambda \neq \lambda^*$, and it is important to determine conditions under which this maximization can be accomplished. If, for the particular choice of λ being used at some cycle, the $P(x,k,\lambda)$ function is unbounded (as a function of x) then the algorithm is not well defined.

II. CONDITIONS FOR A WELL DEFINED ALGORITHM

In this section we give two sets of conditions and prove that under either set of conditions the penalty function has a finite global maximum for any $k > 0 \in E^1$ and any $\lambda > 0 \in E^m$.

Theorem 1: If $f(x) \to -\infty$ as $||x|| \to \infty$ then the algorithm is well defined.

Proof: The proof is simple and requires only that we examine the penalty term for the ith constraint

$$-(1/k) \quad \lambda_i[\exp(kg_i(x))-1] \tag{3}$$

with $k > 0$, $\lambda_i > 0$. As $g_i(x) \to \infty$ (infeasible), (3) $\to -\infty$; but as $g_i(x) \to -\infty$ (feasible) (3) $\to \lambda_i/k$.

Thus,

$$-(1/k) \sum_{i=1}^{m} \lambda_i \exp[(kg_i(x))-1] \leq \sum_{i=1}^{m} \lambda_i/k \tag{4}$$

So if $f(x) \to -\infty$ for all x with $||x|| \to \infty$, then also $P(x,k,\lambda) \to -\infty$ as $||x|| \to \infty$. Hence, P is not unbounded for maximization, and this is true for any $k > 0 \in E^1$ and any $\lambda \geq 0 \in E^m$. □

Theorem 2: If a) $f(x)$ is a strictly concave function

b) $g_i(x)$ are convex functions for all $i = 1,\ldots,m$, and

c) NLP(1) has finite optimal solution x^* with optimal Lagrange multipliers λ^*

then for any fixed $k > 0 \in E^1$, $\lambda > 0 \in E^m$, $\exists$ a finite x which maximizes $P(x,k,\lambda)$.

Proof: For fixed $k > 0$, $\lambda > 0$ we know that $P(x,k,\lambda)$ is a strictly concave function of x because of assumptions a) and b). To show that a finite maximum exists it suffices to show that a finite maximum along any half line exists, that is, $\forall\, x, d \in E^n$, as $\theta \in E^1 \to \infty$ the derivative

$$P'(\theta) = \frac{d}{d\theta} P(x+\theta d,k,\lambda) \tag{5}$$

must eventually become negative (or equal zero).

Since x^* maximizes $P(x,k,\lambda^*)$ (see [2]) we know this is true for $\lambda = \lambda^*$.

Thus, consider any $x, d \in E^n$, $k > 0 \in E^1$, $\lambda > 0 \in E^m$.

$$\begin{aligned} P'(\theta) &= \frac{d}{d\theta} P(x+\theta d,k,\lambda) \\ &= \frac{d}{d\theta} \left\{ f(x+\theta d) - (1/k) \sum_{i=1}^{m} \lambda_i [\exp(kg_i(x+\theta d))-1] \right\} \\ &= f'(\theta) - \sum_{i=1}^{m} \left(\frac{\lambda i}{k}\right) \exp[kg_i(\theta)]\, g_i'(\theta) \end{aligned} \tag{6}$$

where we have introduced the abbreviations $f'(\theta) \equiv \frac{d}{d\theta} f(x+\theta d)$, $g_i(\theta) \equiv g_i(x+\theta d)$, $g_i'(\theta) \equiv \frac{d}{d\theta} g_i(x+\theta d)$.

There are several cases for each of f and g_i :

Cases for f (concave)

f1 $f(\theta) \to -\infty$ as $\theta \to \infty$

f2 $f(\theta)$ is non decreasing as $\theta \to \infty$

then $f'(\theta) > 0$ and decreasing as $\theta \to \infty$ (7)

Cases for each g_i (convex)

g1 $g_i(\theta) \to \infty$ as $\theta \to \infty$

then $\exists r > 0$ such that eventually $g_i'(\theta) \geq r$ so that

$$-\frac{\lambda i}{k} [\exp kg_i(\theta)]\, g_i'(\theta) \to -\infty \text{ as } \theta \to \infty \tag{8}$$

g2 $g_i(\theta)$ is non increasing as $\theta \to \infty$ then $\exp kg_i(\theta) > 0$ and non increasing as $\theta \to \infty$, $g_i'(\theta) \leq 0$ and non decreasing as $\theta \to \infty$, so $-\frac{\lambda i}{k}[\exp kg_i(\theta)]\, g_i'(\theta) \geq 0$ and non increasing as $\theta \to \infty$. (9)

Analysis of Cases

1. If f1 holds, then $P(\theta) \to -\infty$ as $\theta \to \infty$ for any combination of cases on the g_i . (see Theorem 1) Thus, P has a finite global maximum for any $\lambda > 0$.

2. If f2 holds,

2a) If also $\exists i$ with g_i in case g1 then (7), (8), (9) show that $P'(\theta)$ in (6) approaches $-\infty$ as $\theta \to \infty$.

Thus $P'(\theta)$ must eventually become negative so that

P has a finite max along the half line.

2b) Otherwise for all $i = 1,\ldots,m$ constraint g_i is in case <u>g2</u>.

then (6), (7) and (9) show that $P'(\theta) > 0$ as $\theta \to \infty$ so that no maximum is attained. But this is true for any λ including $\lambda = \lambda^*$.

Thus, $P(x,k,\lambda^*)$ does not have a finite maximum contrary to assumption. Hence case 2b cannot occur if NLP has a finite optimal solution. $\square$.

III. SUMMARY

Two sets of conditions which guarantee that the exact penalty function algorithm in [1] is well defined have been presented and proved. It should perhaps also be emphasized that the algorithm in [1] is primarily a method for locating local solutions to NLP (1), and that it can frequently do this even if $P(x,k,\lambda)$ is globally unbounded, (for non concave problems) since in the region of search $P(x,k,\lambda)$ will still have a local maximum. In such a case, however, the possibility of straying into the global unbounded region cannot be overlooked.

REFERENCES

[1] J. K. Hartman, "Iterative Determination of Parameters for an exact Penalty Function". Naval Postgraduate School Technical Report NPS55HH71121A, Dec. 1971.

[2] F. J. Gould and S. Howe, "A New Result on Interpreting Lagrange Multipliers as Dual Variables". University of North Carolina Institute of Statistics Mimeo Series No. 738, January 1971.

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